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Non-Archimedian Fields. Topological Properties of Z_p, Q_p (*p*-adics Numbers)

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Abstract

The present work tries to offer a new approach to some non-Archimedian norms and to some unusual properties determined by these. Then, as an example, p-adic numbers $\mathbf{Q}_p$ *and some topological properties will be illustrated. The present article will present as an example some abstract notions with p-adic numbers extended to* $\mathbf{Q}_p$ *during the translation from non- Archimedian norms to the* $\mathbf{Q}_p$ *norm.*

Key words: *p-adics, non-archimedian, topological properties*

Non-Archimedian Absolute Value and Non-Archimedian's Properties Fields

Definition 1. A function $|\cdot| : \mathbf{E} \to \mathbf{R}^+$ on a field **E** satisfying:

$$(1)\ |x| = 0 \Leftrightarrow x = 0$$
$$(2)\ |xy| = |x||y|$$
$$(3)\ |x + y| \le |x| + |y| \quad (\textit{the Triangle Inequality})$$

is called a norm or an absolute value.

Definition 2. Let **E** be a fields with an absolute value $|\cdot|$.

Let $a \in \mathbf{E}$, $r \in \mathbf{R}^+$. The open ball of radius *r* centered at *a* is $B_r(a) = \{x \in \mathbf{E} : |x - a| < r\}$.

A closen ball of radius *r* centered at *a* is $B_r(a) = \{x \in \mathbf{E} : |x - a| \le r\}$.

A clopen ball of radius *r* centered at *a* is an open and in the same time an closen, ball.

Definition 3. An absolute value is called non-archimedian if it satisfies a stronger version of the Triangle Inequality:

$$|x + y| \le \max\{|x|, |y|\}\ \forall x, y \in \mathbf{E}$$

and archimedian otherwise; and a field **E** with an non-archimedian absolute value is called a non-archimedian field.

Proposition 1. An absolute value $|\cdot|$ on **Q** is non-archimedian if and only if $|x| \le 1\ \forall x \in \mathrm{N}$.

Proof. First suppose $|\cdot|$ is non-archimedian.

Then for introduction $n \in \mathbf{N}$, let be *'P(n) ':* $|n| \le 1$.

For $n = 1$ we have $|1| = 1 \le 1$; $P(1)$ is true.

Considering now $|n| \le 1$ *for* $n = k$. Then $|k + 1| \le max\ \{|k|, 1\} = 1 \le 1$ and $P(k) \Rightarrow P(k + 1)$.

Trough induction, $|n| \le 1,\ \forall n \in \mathbf{N}$.

Now we suppose $|n| \le 1,\ \forall n \in \mathbf{N}$ and we want to show $|a + b| \le max\ \{|a|, |b|\},\ \forall \mathrm{a}, \mathrm{b} \in \mathbf{Q}$.

If $b = 0$ we have $|a + b| = |a + 0| = |a| = max\{|a|, |0|\} \le max\ \{|a|, |b|\}$.

So if we assume that $b \neq 0$ then

$$|a+b| \le \max\{|a|,|b|\} / \frac{1}{|b|} \Leftrightarrow \left|\frac{a+b}{b}\right| \le \max\left\{\frac{|a|}{|b|};1\right\} \Leftrightarrow \left|\frac{a}{b}+1\right| \le \max\left\{\left|\frac{a}{b}\right|;1\right\}$$

and then it is enough to show that $|x + 1| \le max\ \{|x|, 1\}, \forall \mathrm{x} \in \mathbf{Q}$.

For that, let be:

$$|x+1|^n = \left|\sum_{k=0}^{n} C_n^{\ k} x^k\right| \le \sum_{k=0}^{n} \left|C_n^{\ k}\right| |x|^k \le \sum_{k=0}^{n} |x|^k .$$

If $|x| \le 1$ then $|x|^k \le 1$ for $k = 1,2, \ldots ,n$.

If $|x| > 1$ then $|x^k| \le |x^n|$ for $k = 1,2, \ldots ,n$.

And in both cases, $\sum_{k=0}^{n} |x|^k \le (n+1)\cdot \max\{1, |x|^n\}$.

Then $|x+1|^n \le \sum_{k=0}^{n} |x|^k \le (n+1)\cdot \max\{1,|x|^n\}$ and $|x+1| \le \sqrt[n]{(n+1)}\max\{1,|x|\}$

and because $\sqrt[n]{n+1} \to 1$ *when* $n \to \infty$ we have for the last relations:

$$|x + 1| \le max\ \{|x|, 1\},$$

what we want to show.

Also another result which is known is that: an absolute value $|\cdot|$ on $\mathbf{Q}$ is archimedian if $\forall \mathrm{x}, \mathrm{y} \in \mathbf{Q}$, $(\exists)$ $n \in \mathbf{N}$ such $|nx| > |y|$.

Lemma. In a non-archimedian field **E** if $x, y \in \mathbf{E}$, $|x| < |y|$, then $|y| = |x + y|$.

Proof.

$$\left.\begin{array}{l} Assume\ \ |x| < |y| \\ |x+y| \le \max\{|x|,|y|\}\ by\ the\ initial\ propreties \end{array}\right\} \Rightarrow |x+y| \le |y|. \qquad (1)$$

Also, $y = (x + y) - x$ we have $|y| \le max\ \{|x + y|, |-x|\} = max\ \{|x + y|, |x|\}$ and because $|x| < |y|$ for the last relations $\Rightarrow$

$$|y| \le |x + y|. \qquad (2)$$

Now for (1) and (2) we have $|y| = |x + y|$.

Proposition 2. In a non-archimedian field **E** every "triangle" is isosceles.

Proof. Let x, y, z be the "verticus" of the triangle an $|x-y|$, $|y-z|$, $|x-z|$.

If $|x-y| = |y-z|$ we are done.

If $|x-y| \neq |y-z|$ then we assume $|y\text{-}z| < |x\text{-}y|$.

But by Lemma, because $|y\text{-}z| < |x\text{-}y|$, we have $|x-y| = |(x-y)+(y-z)| = |x-z| \Rightarrow |x-y| = |x-z|$ and the triangle is isosceles.

Proposition 3. In a non-archimedian field. **E**, every point in an open ball is a center and $b \in B_r(a) \Rightarrow B_r(a) = B_r(b)$.

Proof. Let $b \in B_r(a)$, so $|b-a| < r$ and x any element of $B_r(a)$.

$|x-b| = |(x-a)+(a-b)| \leq max\ \{|x-a|, |a-b|\} < r.$

Hence, $x \in B_r(b)$ since $|x-b| < r \Rightarrow B_r(a) \subseteq B_r(b)$.

Same $B_r(b) \subseteq B_r(a)$ is obviously now.

Therefore, $B_r(a) = B_r(b)$.

Corollary 1. Let **E** be a non-archimedian field. Then for two any open balls or one is contained in the other, or is either disjoint.

Proof. We assume $p < r$ and the problem is:

$$x \in B_p(a) \text{ and } x \in B_r(a) \Rightarrow B_p(a) \subseteq B_r(b) \text{ or } B_p(b) \subseteq B_r(a).$$

By Proposition 3, we have**:**

$$x \in B_r(a) \Rightarrow B_r(a) = B_r(x) \text{ and } x \in B_r(b) \Rightarrow B_r(b) = B_r(x).$$

Then $x \in B_p(a) = B_p(x) \subseteq B_r(x) = B_r(b)$ since $p < r$.

So, $B_p(a) \subseteq B_r(b)$.

Corollary 2. In a non-archimedian field every open ball is clopen, that is a ball which is open and closed.

Proof. Let $B_p(a)$ be an open ball in a non-archimedian field.

Take any x in the boundary of $B_p(a) \Leftrightarrow B_r(x) \cap B_p(a) \neq 0$ for any $s > 0$, so in particular, for s < r.

By Corollary 1, since $B_r(x)$ and $B_p(a)$ are not disjoint, one is conteined in the other and because $s < r$ we have $x \in B_r(x) \subseteq B_p(a) \Rightarrow B_p(a)$ is closed.

So every open ball is closed and every ball is clopen.

The *p*-adic Absolute Value and Topological Properties for Z_p and Q_p Non-Archimedians Fields

Definitions. Let $p \in$ **N** be a prime, then for each $n \in$ **Z,** $n \neq 0$ we have $n = p^{\alpha} n'$ with p not divides n'.

We define $f_p(n) = \begin{cases} \infty & for\ n = 0 \\ \alpha & when\ n \neq 0 \end{cases}$ and $|n|_p = p^{-f_p(n)}$ is a non-archimedian absolute value on **Z**, called the p-adic absolute value.

For $\frac{a}{b} \in \mathbf{Q}$, $f_p\left(\frac{a}{b}\right) = f_p(a) - f_p(b)$ we define a non-archimedian p-adic absolute value in $\mathbf{Q}$ with the relation:

$$|x|_p = p^{-f_p(x)}, \ (\forall)\ x \in \mathbf{Q}$$

and

$$|x - y|_p = p^{-f_p(x-y)} \ (\forall)\ x, y \in \mathbf{Q}.$$

Then we have:

$$\mathbf{N}_p = \left\{\alpha / \alpha = \sum_{i=0}^{n} a_i \cdot p^i, a_i, n \in \mathbf{N}; 0 \le a_i \le p-1\right\}$$

are the p-adic numbers from **N,**

$$\mathbf{Z}_p = \left\{\alpha / \alpha = \sum_{i=0}^{\infty} a_i \cdot p^i, a_i \in \mathbf{N}; 0 \le a_i \le p-1\right\}$$

are the p-adic numbers from **Z** and we can define these numbers as the inverses of naturals numbers written in p-base system and the naturals numbers.

$$\mathbf{Q}_p = \left\{\alpha / \alpha = \sum_{i=k}^{\infty} a_i \cdot p^i, k \in \mathbf{Z}, a_i \in \mathbf{N}; 0 \le a_i \le p_i - 1\right\}$$

are the p-adic numbers from **Q**.

Proposition 4. If **E'** it's the completion of a field **E** with respect to an absolute value |.| then **E'** it's a field where we can to extend the absolute value |.|

Proof. See [3] of references.

Proposition 5. $\mathbf{Q}_p$ is the completion of **Q** with respect to $|\cdot|_p$.

Proof. See [3] of references.

Proposition 6. $\mathbf{Z}_p$,$\mathbf{Q}_p$ are completes.

Proof. See [3] of references.

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p \ / \ |x|_p \le 1\}.$$

Proof. See [5] of references.

Proposition 7. $\mathbf{Z}_p$ is compact and $\mathbf{Q}_p$ is locally compact, by which we mean every $x \in \mathbf{Q}_p$ have a neighborhood which is compact.

Proof.

(1) We know that $\mathbf{Z}_p$ is complete, we want to show that $\mathbf{Z}_p$ is totally bounded.

Take any $\varepsilon > 0$ and $n \in \mathbf{N}$ such that $p^{-n} \le \varepsilon$..

Then $a + p^n \cdot \mathbf{Z}_p = B_{p^{-n}}(a) \subseteq B_\varepsilon(a)$ because $|x - a|_p \le p^{-n} < \varepsilon \ (\forall)\ x \in a + p^n\mathbf{Z}_p$.

For $a \in \mathbf{Z} / p^n\mathbf{Z}$ the classes reprezentants is included in $\{0, 1, \ldots, p, \ldots, p^n - 1\}$.

We have with definition of $\mathbf{Z}_p$

$$\mathbf{Z}_p \subset \bigcup_{a\in Z/p^nZ} a + p^n\mathbf{Z}_p = \bigcup_{a\in Z/p^nZ} B_{p^{-n}}(a) \subseteq \bigcup_{a\in Z/p^nZ} B_\varepsilon(a)$$

a finite cover of $\mathbf{Z}_p$.

So $\{B_\varepsilon(a) : a \in \mathbf{Z} / p^n\mathbf{Z}\}$ is a finite set of open balls which cover $\mathbf{Z}_p$, so $\mathbf{Z}_p$ is totally bounded, also is complete and result it is compact.

(2) Let be $f : \mathbf{Z}_p \to \mathbf{Q}_p; f(y) = x + y$ for $x \in \mathbf{Q}_p$ known , is continuous.

$\mathbf{Z}_p$ is compact so the image of $\mathbf{Z}_p$ is $x + \mathbf{Z}_p$ is also compact.

Now we can observe that $x + \mathbf{Z}_p$ is a compact neighborhood of $x \in \mathbf{Q}_p$ because for $z \in x + \mathbf{Z}_p; z - x \in \mathbf{Z}_p; |z - x|_p < 1$, (Proposition 2).

Remarks

We give, for students, very common definitions, available in spaces with a norm.

Definitions

Let (x_n) be a sequence with entries in a field **E** with an absolute value |.|.

1. (x_n) is a Cauchy sequence if

$$(\forall)\ \varepsilon > 0, (\exists)\ N_\varepsilon \in N$$

such that $(\forall)\ m, n \geq N, | x_m - x_n | < \varepsilon$.

2. If every Cauchy sequence in **E** converges, **E** is complete with respect to |.|.
3. We can introduce the equivalence relation:

$$(x_n) \equiv (y_n) \text{ if } (\forall)\ \varepsilon > 0, (\exists)\ N_\varepsilon \in N$$

such that $(\forall)\ n \geq N_\varepsilon, | x_n - y_n | < \varepsilon.$

4. The completion of field **E** with respect to an absolute value |.| is the set :

$\{[(x_n)]$: (x_n) is a Cauchy sequences on **E**$\}$ of all equivalence classes of Cauchy sequence with the equivalence relation from 4).

5.

   5.1. An open cover of a set S is a family of open sets $\{S_i\}$ such that $\cup_i S_i \supseteq S$.

   5.2. A set S is called compact if every open cover of S has a finite subcover. If $\{S_i\}$ is any open cover of $\cup_{i=1^n}^n S_i \supseteq S$ for some $n \in \mathrm{N}$.

6. A set is called totally bounded if for ever ε > 0, there exist a finite collection of balls of radius ε which cover the set.

Propositions

1. Compactness is preserved by continuous functions, what it means, if f is continuous and A is compact then $f(A)$ is compact.
2. Any closed subset of a complete space is complete.
3. A set is compact if it is complete and totally bounded.

References

1. Collingwood, J., Marion, S., Mazowita, M. - *P-adic number*, http://padic.mathstat.uottawa.ca/~MAT3166/reports/p-adic.pdf
2. Colojoara, I. - *Analiză matematică*, Ed. Didactică şi Pedagogică, Bucureşti, 1983

3. G o u v e a , F . Q . - *P-adic Numbers: An introduction*, Second Edition, Springer-Verlag, New York, 1997
4. K a t o k , S . - Real and p-adic analysis, *Course notes for math 497c Mass program,* fall 2000, Revised November 2001, Department of mathematics, the Pennsylvania State University
5. K o b l i t z , N . - *P-adic numbers, P-adic analysis and Z-eta Functions,* Second edition, Springer-Verlag, New York, 1984

Câmpuri nearhimediene. Proprietăţi topologice ale lui $\mathbf{Z}_p$, $\mathbf{Q}_p$ (numere *p*-adice)

Rezumat

Lucrarea doreşte să dea o nouă abordare asupra unor norme nearhimediene şi asupra unor ciudate proprietăţi induse de acestea, apoi, ca exemplu, sunt prezentate numerele p-adice $\mathbf{Q}_p$ *şi câteva proprietăţi topologice. Trecând de la norme nearhimediene la norma în* $\mathbf{Q}_p$ *se va da o exemplificare a unor noţiuni abstracte în numerele din baza p extinse la* $\mathbf{Q}_p$.